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The Hierarchical and Social Decision Making Procedure for Bi-Objective Optimization Problems by Using the Simple Majority Decision Rule

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Abstract. Multi-objective optimization has been successfully applied to problems of industrial design, problems of quality control and production management, and problems of finance. The theme of these applications is how to choose the best solution for the decision makers out of a set of non-inferior solutions to a multi-objective optimization problem. For this purpose, an optimization model with hierarchical structure, whose lower problem is a multi-objective optimization problem and the upper problem is a preference optimization problem on a set of non-inferior solutions, must be constructed. This kind of hierarchical problems have been previously analyzed only with regard to linear programming problems by Benson[6]. In this paper, an algorithm is derived that provides a solution as a social choice, obtained by aggregating plural decision-makers' preferences. In the case of the simple majority rule, the bi-objective problem is transformed into an ϵ-parameter choice problem, and the golden section method is applied. The availability of the approach is demonstrated with the means of an illustrative example.

Keywords: Hierarchical Decision Making, Social Preference, Bi-Objective Optimization, Simple Majority Decision Rule, Aggregation of Preferences, ϵ-Constraint Method, Non-Inferior Solution Set, Golden Section Method

1. Introduction

The research regarding multi-objective optimization problems has been developed from the mathematical level[1],[2] to the application to practical problems. As you can see in the literatures [3]~[5] the research is successfully applied to problems of industrial design like structure designing, problems of management like quality control and production management, and problems of finance. The theme of the application of this type of multi-objective optimization is how to choose the best solution for the decision makers out of a set of non-inferior solutions of a multi-objective optimization problem.

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For this theme, after identifying the decision makers' preference functions, an optimization model with a hierarchical structure, whose lower problem is a multi-objective optimization problem and whose upper problem is a preference optimization problem on the set of non-inferior solutions, must be constructed. This kind of hierarchical problems are analyzed only regarding linear programming problems in the reference [6]. However analysis of non-linear problems is not easy, and the decision makers' preference functions concerning non-inferior solutions are actually difficult to identify. Therefore many man-machine interactive preference optimization methods[7]~[12], which find their best non-inferior solution by partially getting some information from decision makers without identifying preferences or goals of decision makers, are proposed and are actually used to have an implementable solution. In these methods, however, the existence of preferences or goals of a single decision maker is premised.

Methods are hardly proposed, in which preferences or goals of plural agents having decision making ability are assumed, and which find the best non-inferior solution for a group by aggregating these preferences or goals, as for example in political decision making problems with participation of inhabitants in the field of public policy etc. Recently the groupware is often talked about as a problem solving which employs the environment of computer networks in the field of information processing, and especially CSCW (Computer Supported Cooperative Work) is the central research theme. However the main research about it is regarding the support environment by computers in terms of hard- and software. It is actually rare that a process of group decision making is considered as an algorithm interactively performed by man-machine and that the convergence etc. of its process are mathematically and numerically examined based on the utility theory or the preference theory etc. [13]. We believe that the result of this research can offer valid methods, which are based on relatively theoretical background.

In the procedure to be presented, bi-objective function values corresponding to some trial solutions picked up from a non-inferior solution set are ordered by plural decision-makers' appreciation based on their individual preferences, and a social preference with respect to these experimental values is generated by aggregating these individual information with the simple majority decision rule. Then a new non-inferior solution set is produced based on these experimental results, and it is possible to generate better social preference from this. Such iterative processes reach at the best non-inferior solution as the social choice.

The procedure to pick up experimental non-inferior solutions and to calculate their corresponding non-inferior objective values as alternatives is done by the ε-constraint method[9],[16] which is one of scalarization methods for multi-objective optimization problems. Therefore a problem to require the best non-inferior solution as a social choice can be transposed into a problem to choose the best parameter ε in the ε-constraint problem based on the social preference. On the other hand, properties of ordering by the social preference based on the simple majority decision rule limit the problem with bi-objective cases where the ε-parameter becomes a scalar. In this case, linear search methods are applicable for searching the best

ε-parameter as the social choice. The golden section method[18] is used as a linear search method in our proposed method.

2. Formulation of the Hierarchical and Social Decision Making Problems for Bi-objective Optimization

The bi-objective optimization problem discussed in this paper is formulated as

$$\min \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix} \tag{1}$$

$$\text{subj. to } \boldsymbol{x} \in X \subset R^n,$$

where $f_i, i = 1,2$ is defined on the n-dimensional real space R^n,and $\boldsymbol{f}(\boldsymbol{x})$ is represented as

$$\boldsymbol{f}(\boldsymbol{x})^T = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$$

to simplify description. Then the non-inferior solution set, which are the mathematical rationality of this problem, is presented as

$$\hat{X} = \{\hat{\boldsymbol{x}} \in X \mid \nexists\, \boldsymbol{x} \in X \text{ such that } f(\boldsymbol{x}) \leq f(\hat{\boldsymbol{x}})\}$$

and the set of objective function values corresponding to this set is called a non-inferior value set and is represented as

$$\hat{F} = \{f(\hat{\boldsymbol{x}}) \mid \hat{\boldsymbol{x}} \in \hat{X}\}.$$

When the unique non-inferior value and its corresponding non-inferior solution are chosen from this non-inferior value set for the bi-objective problem in the lower level, a problem to have a social choice by constructing a kind of social preference through aggregation of the preferences of plural decision-makers in the upper level is called a social decision making problem for the bi-objective problem. The question is 'how to aggregate preferences of plural decision-makers and construct the social preference'.

In this paper, the simple majority decision rule, which is generally used in the field of sociology, management, and politics, is taken as a rule to construct the social choice. More generally, the social welfare function, which satisfies the Arrow's conditions[14] as much as possible, may be adopted as a rational rule to construct the social preference. However it is known as the Arrow's impossibility theorem[14] that a social welfare function which satisfies all of the Arrow's conditions does not exist. Therefore a weak order relation♣ is especially required on the social preference in order to enable to choose the most preferable alternative based on the social choice.

Let the number of decision-makers be m, and all the decision-makers have their own individual preference relation on the set of bi-objective values. The preference relations cannot be represented explicitly, however, the relations are corresponding to the relations of large and small values of the implicit preference functions. Hence, on the alternative set Y with a proper size including bi-objective function value set $\{ \boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in R^n \}$, the m decision-makers' preference functions $\phi_1, \cdots, \phi_m$ are defined implicitly and the decision-maker i's individual preference $(Y, \succsim_i)$ is defined as follows by the preference function ϕ_i's value.

$$\begin{aligned}
&\boldsymbol{y}^1 \succ_i \boldsymbol{y}^2 \Leftrightarrow \phi_i(\boldsymbol{y}^1) < \phi_i(\boldsymbol{y}^2) \\
&\boldsymbol{y}^1 \sim_i \boldsymbol{y}^2 \Leftrightarrow \phi_i(\boldsymbol{y}^1) = \phi_i(\boldsymbol{y}^2) \qquad (2) \\
&\boldsymbol{y}^1 \succsim_i \boldsymbol{y}^2 \Leftrightarrow \phi_i(\boldsymbol{y}^1) \leq \phi_i(\boldsymbol{y}^2)
\end{aligned}$$

In this case, let a smaller value of ϕ_i be more preferable taking into account the fact that problem(1) is a minimization problem.

By the way, if an individual preference $(Y, \succsim_i)$ takes every possible form, a preference function ϕ_i which gives an individual preference takes every possible form of functions. Let a set of these functions ϕ_i be Φ_i. Moreover, let a preference function which gives a social preference $(Y, \succsim)$ constructed by a certain rule from m individual preferences $(Y, \succsim_i), i = 1, \cdots, m$ be ϕ, and a set of every possible social preference functions ϕ be Φ. In this case, a social welfare function which gives a social preference from m individual preferences is a mapping from the individual preference functions' direct product set $\Phi_1 \times \cdots \times \Phi_p$ into the social preference functions' set, therefore it is an operator that generates a function from functions, and which is represented by Ψ. If the m individual preference functions $\phi_1, \cdots, \phi_m$ are given explicitly, the social preference function ϕ is represented by

$$\phi = \Psi(\phi_1, \cdots, \phi_m),$$

as a result of the fact that the operator Ψ is acting on m functions $(\phi_1, \cdots, \phi_m)$. $\phi_i, i = 1, \cdots, m$ are functions defined on the set Y, and ϕ is also a function on the set Y. To express this matter clearly, the expression below is used.

$$\phi(\boldsymbol{y}) = \Psi(\phi_1(\cdot), \cdots, \phi_m(\cdot))(\boldsymbol{y})$$

Here $\phi_i(\cdot)$ indicates a functional form defined on the set Y. One functional form $\phi(\cdot)$ is decided by operating a social welfare function Ψ on the individual functions $\phi_1(\cdot), \cdots, \phi_m(\cdot)$, and the value of the function ϕ is decided as ϕ $(\boldsymbol{y})$ for $\boldsymbol{y}$ on the set Y.

With the preparation above in mind, the social decision making problem for the bi-objective problem is formulated as below.

$$\min_{\hat{\boldsymbol{x}}} \phi(\mathbf{f}(\hat{\boldsymbol{x}})) = \Psi(\phi_1(\cdot), \cdots, \phi_m(\cdot))(\mathbf{f}(\hat{\boldsymbol{x}})) \tag{3a}$$

$$\text{subj. to } \hat{\boldsymbol{x}} \in \hat{X},$$

or in a hierarchical form companied with the bi-objective optimization problem(1):

$$\min_{\hat{\boldsymbol{x}}} \phi(\boldsymbol{f}\ (\hat{\boldsymbol{x}})) = \Psi(\phi_1(\cdot), \cdots, \phi_m(\cdot))(\boldsymbol{f}(\hat{\boldsymbol{x}})) \tag{3b}$$

$$\text{subj. to } \hat{\boldsymbol{x}} \ = \arg\min_{\boldsymbol{x}} \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix}$$

$$\text{subj. to } \boldsymbol{x} \in X.$$

The difficulties associated with this problem are summarized in the following points.

1. Even if decision-makers subconsciously have their own preference relations based on their individual preference functions, forms of the functions cannot be given explicitly.

2. Even supposing that forms of individual preference functions $\phi_1, \cdots, \phi_m$ are given explicitly, it is impossible in fact to pick up every alternative from the alternative set Y, to operate a certain social welfare function Ψ (for instance, the simple majority decision rule), and to predetermine a social preference function by the above procedures on the set Y.

3. It is also difficult to give the set $\hat{X}$ of the non-inferior solutions, or its corresponding non-inferior value set $\hat{F}$ explicitly in advance.

In order to conquer these difficulties, an iterative search method supporting social decision making is offered in following section.

3. An Iterative Method Supporting the Social Decision Making for Bi-objective Problems

3.1. ε-constraint method

First of all, a solution of the obstacle 3) in the previous section is found with application of the ε-constraint method. The ε-constraint method concerned with the bi-objective problem (1) is to solve the ε-constraint problem,

$$\min_{\boldsymbol{x}} f_2(\boldsymbol{x}) \tag{4a}$$

$$\text{subj. to } f_1(\boldsymbol{x}) \leqq \varepsilon \tag{4b}$$
$$\boldsymbol{x} \in X. \tag{4c}$$

Let the minimum solution, which depends on ε, be $\boldsymbol{x}^o(\varepsilon)$ and the corresponding minimum valud be $\omega(\varepsilon)$. That is

$$\begin{aligned}\omega(\varepsilon) &= f_2(\boldsymbol{x}^o(\varepsilon)) \\ &= \min\{f_2(\boldsymbol{x}) \mid f_1(\boldsymbol{x}) \leqq \varepsilon,\ \boldsymbol{x} \in X\}.\end{aligned}$$

Then, denote by E_a the region of values for the parameter ε for which the ε-constraint (4.b) becomes active at the solution point $\boldsymbol{x}^o(\varepsilon)$ to the ε-constraint problem;

$$E_a = \{\varepsilon \mid f_1(\boldsymbol{x}^o(\varepsilon)) = \varepsilon\}. \tag{5}$$

It is known that the non-inferior value set of the problem (1) $\hat{F} = \{f(\hat{\boldsymbol{x}}) \mid \hat{\boldsymbol{x}} \in \hat{X}\}$ is given by

$$\hat{F}\{(\varepsilon, \omega(\varepsilon)) \mid \varepsilon \in E_a\} \tag{6}$$

and that the minimum solution $\boldsymbol{x}^o(\varepsilon)$ for $\omega(\varepsilon)$ is the non-inferior solution which gives the non-inferior value $(\varepsilon, \omega(\varepsilon))$.[17] Therefore, replacing the non-inferior value set $\{f(\hat{\boldsymbol{x}}) \mid \hat{\boldsymbol{x}} \in \hat{X}\}$ in problem(3) by the set $\hat{F}$ above, the social choice problem(3) is transformed equivalently into

$$\min_{\varepsilon} \phi(\varepsilon, \omega(\varepsilon)) = \Psi(\phi_1(\cdot), \cdots, \phi_m(\cdot))(\varepsilon, \omega(\varepsilon)) \tag{7a}$$

$$\text{subj. to } \varepsilon \in E_a,$$

or in two-level form

$$\min_{\varepsilon} \phi(\varepsilon, \omega(\varepsilon)) = \Psi(\phi_1(\cdot), \cdots, \phi_m(\cdot))(\varepsilon, \omega(\varepsilon))$$

$$\text{subj. to } f_1(\boldsymbol{x}^o(\varepsilon)) = \varepsilon \tag{7b}$$

$$\boldsymbol{x}^o(\varepsilon) = \arg\min_{\boldsymbol{x}} f_2(\boldsymbol{x})$$

$$\text{subj. to } f_1(\boldsymbol{x}) \leqq \varepsilon$$

$$\boldsymbol{x} \in X.$$

It is not easy to give the set E_a explicitly, however, it is possible to calculate all the non-inferior values from $(\varepsilon, \omega(\varepsilon))$ with the parameter ε ranging over the values in the set E_a. Moreover if ε, which minimizes the social preference function $\phi(\varepsilon, \omega(\varepsilon))$,

is found in the set E_a, the corresponding solution $\boldsymbol{x}^{\,o}(\varepsilon)$ to ε-constraint problem(4) becomes the best non-inferior solution for the social preference, i.e., it becomes the social choice for the bi-objective problem. Problem(7) is called 'a social choice problem for a bi-objective problem by the ε-constraint method'.
Here, let the following assumption be imposed.

(**Assumption 1**) The vector objective function $\boldsymbol{f} = (f_1, f_2)^T$ is convex, and the set X is convex.

Then, the non-inferior values set $\hat{F}$ is expressed as part of a continuous line which is convex in to the left and below. Therefore E_a, the set of values for the parameter ε that satisfy $(\varepsilon, \omega(\varepsilon)) \in \hat{F}$, is a closed interval on the f_1-axis. Its lower limit $\varepsilon_{\min}$ is the minimum value which is obtained at $\boldsymbol{x}'$ by minimizing f_1 singly;

$$\varepsilon_{\min} = f_1(\boldsymbol{x}') = \min\{f_1(\boldsymbol{x}) \mid \boldsymbol{x} \in X\}. \tag{8}$$

And its upper limit value $\varepsilon_{\max}$ is given by

$$\varepsilon_{\max} = f_1(\boldsymbol{x}''), \tag{9a}$$

where $\boldsymbol{x}''$ is the solution to the problem of minimizing f_2 singly;

$$f_2(\boldsymbol{x}'') = \min f_2(\boldsymbol{x}) \quad \text{subj. to } \boldsymbol{x} \in X. \tag{9b}$$

That is, E_a is given explicitly as

$$E_a = \{\varepsilon \mid f_1(\boldsymbol{x}') \leqq \varepsilon \leqq f_1(\boldsymbol{x}'')\}.$$

Hence, in conclusion, the problem(7) can be represented as follows.

$$\min_{\varepsilon} \phi(\varepsilon, \omega(\varepsilon)) = \Psi(\phi_1(\cdot), \cdots, \phi_m(\cdot))(\varepsilon, \omega(\varepsilon)) \tag{10a}$$

$$\text{subj. to } f_1(\boldsymbol{x}') \leqq \varepsilon \leqq f_1(\boldsymbol{x}'') \tag{10b}$$

where

$$\omega(\varepsilon) = \min\{f_2(\boldsymbol{x}) \mid f_1(\boldsymbol{x}) \leqq \varepsilon,\ \boldsymbol{x} \in X\}$$

$$\boldsymbol{x}' = \arg\min\{f_1(\boldsymbol{x}) \mid \boldsymbol{x} \in X\}$$

$$\boldsymbol{x}'' = \arg\min\{f_2(\boldsymbol{x}) \mid \boldsymbol{x} \in X\}$$

3.2. The simple majority decision rule

Solutions of the obstacle 1) and 2) are found by using the simple majority decision rule as a social welfare function Ψ on the condition that each decision-maker has an individual preference function subconsciously. In this case, if two alternatives $\boldsymbol{y}^1, \boldsymbol{y}^2$ among the objective values are shown, each decision-maker can express his individual preference $\succ_i$ for the two alternatives by formula(2). Hence if two kinds of parameter $\varepsilon^1 \in E_a$, $\varepsilon^2 \in E_a$ are given as the ε-parameter, non-inferior values $\boldsymbol{y}^1 = (\varepsilon^1, \omega(\varepsilon^1)), \boldsymbol{y}^2 = (\varepsilon^2, \omega(\varepsilon^2))$ can be shown to the decision-makers by solving the ε-constraint problems corresponding to ε^1 and ε^2, and each decision-maker is able to express

$$\boldsymbol{y}^1 \succ_i \boldsymbol{y}^2,\ \boldsymbol{y}^2 \succ_i \boldsymbol{y}^1,\ \text{or}\ \boldsymbol{y}^1 \sim_i \boldsymbol{y}^2,$$

based on his subconscious individual preference relations. This access is repeated to all of m decision-makers in order to aggregate informations of these individual preferences, and then the social preference relation between $\boldsymbol{y}^1$ and $\boldsymbol{y}^2$ is decided among relations of

$$\boldsymbol{y}^1 \succ \boldsymbol{y}^2,\ \boldsymbol{y}^2 \succ \boldsymbol{y}^1,\ \text{or}\ \boldsymbol{y}^1 \sim \boldsymbol{y}^2$$

based on the simple majority decision rule.

To introduce a social preference relation on the non-inferior value set by taking the above procedures on every pair $(\boldsymbol{y}^1, \boldsymbol{y}^2)$ on the non-inferior set and to order all the non-inferior values based on this social preference relation in advance requires the construction of a social preference function ϕ to $(\varepsilon, \omega(\varepsilon))$, where ε is located in the set E_a of problem (7). Hence express it as $\tilde{\phi}(\varepsilon) = \phi(\varepsilon, \omega(\varepsilon))$ for convenience, problem (7) can be solved by finding ε^o which minimizes $\tilde{\phi}(\varepsilon)$. The pair $(\varepsilon^o, \omega(\varepsilon^o))$ of $\varepsilon^o \in E_a$ and the minimum value of the ε-constraint problem $\omega(\varepsilon^o)$ is the social choice on the non-inferior value set of the bi-objective optimization problem(1).

By the way, a question here is whether all non-inferior values can be ordered by social preference relations based on the simple majority decision rule or not. In conclusion, in the case that decision-makers' preferences are single peaked and the objects to be evaluated are expressed by scalar values, it is known that social preferences based on the simple majority decision rule are transitive and ordering all the alternatives among objective values is possible[14]. for the case that the objects are expressed by vector values, however, it is known that the transitivity does not hold for social preferences based on the simple majority decision rule, although the decision-maker's preferences may still be single peaked, and therefore ordering all the alternatives for these objects is impossible. Social choice problems for the bi-objective problems considered in this paper are transformed into social choice problems of the scalar parameter ε subject to ε-constraint problems as in (10). Therefore impose the following assumption:

(Assumption 2) All the preferences of m decision-makers are single peaked on the non-inferior value set (the non-inferior curve in the bi-objective case).

Then, the transitivity holds on the social preference relations concerning ε, and it is possible to order all the ε-parameters by social preferences.

Such social choice problems for bi-objective problems are themselves special cases of social choice problems for multi-objective problems. To solve problems which have two conflicting purposes such as 'the economical development' and 'the conservation of the environment', however, is a realistic and practical as well as basic matter to settle various kinds of economical and social conflicts.

3.3. A search method for the social choice by the golden section method

Here the method for the actual solution of problem(10) is discussed. If two arbitrary parameters $\varepsilon_1, \varepsilon_2 \in E_a$ are chosen among the set $E_a = \{\varepsilon \mid f_1(\boldsymbol{x}') \leqq \varepsilon \leqq f_1(\boldsymbol{x}'')\}$, the minimum values $\omega(\varepsilon_1), \omega(\varepsilon_2)$ of their corresponding ε-constraint problem (4a,b,c) are calculated, and the two non-inferior values $(\varepsilon_1, \omega(\varepsilon_1)), (\varepsilon_2, \omega(\varepsilon_2))$ are suggested to the decision-makers, the social preference between these two alternatives can be decided by the simple majority decision rule from the decision-maker's individual preference relation. Hence if this procedure has been done in advance for every pair of ε-parameters, the ε-parameters are ordered by the social preference composed by the former procedure, the parameter ε which gives the best social preference is chosen as the social choice, and a solution of its corresponding ε-constraint problem is chosen as the non-inferior solution of the social choice. However, it is not efficient to compose the social preference in advance by comparison of all pairs. This paper suggests an algorithm which repeats the following procedures;

- Order only two experimentally chosen ε-parameters based on decision-maker's individual preferences,
- Eliminate the area, on which no parameter ε corresponding to the social choice exists, by the individual preference information,
- Generate a new preferable ε-parameter from the area on which parameter ε corresponding to the social choice is expected to exist,
- Order the new ε-parameter among the chosen parameters.

Here assuming that problem (10) is of minimization type with respect to the scalar quantity ε, the golden section method, one of the most efficient linear search methods, is applied to solve it. Hence, let the composite function $\phi(\varepsilon, \omega(\varepsilon))$ be $\tilde{\phi}(\varepsilon) = \phi(\varepsilon, \omega(\varepsilon))$. Under Assumption 2, it is known that the social choice is also single peaked. Therefore it is expected that the social function $\tilde{\phi}$, which gives the social preference relation concerning ε, is a convex-like function although it is not expressed explicitly. Suppose that the social choice parameter ε, which minimizes $\tilde{\phi}(\varepsilon)$, exists on the experimentally chosen section $[\varepsilon^a(k), \varepsilon^b(k)]$, and that $\varepsilon^a(k), \varepsilon^u(k), \varepsilon^v(k)$ and $\varepsilon^b(k)$ are picked up as

$$\varepsilon^a(k) < \varepsilon^u(k) < \varepsilon^v(k) < \varepsilon^b(k),$$

where k shows the number of iterations.

(i) If $\varepsilon^u(k) \precsim \varepsilon^v(k)$, then $\varepsilon \precsim \varepsilon^v(k)$ for all $\varepsilon \in [\varepsilon^a(k), \varepsilon^u(k)]$, and the social choice parameter ε does not exist on $[\varepsilon^a(k), \varepsilon^u(k)]$ but on $[\varepsilon^u(k), \varepsilon^b(k)]$.

(ii) If $\varepsilon^u(k) \succsim \varepsilon^v(k)$, then $\varepsilon \precsim \varepsilon^u(k)$ for all $\varepsilon \in [\varepsilon^v(k), \varepsilon^b(k)]$, and the social choice parameter ε dose not exist on $[\varepsilon^v(k), \varepsilon^b(k)]$ but on $[\varepsilon^a(k), \varepsilon^v(k)]$.

Considering this fact, new smaller section $[\varepsilon^a(k+1), \varepsilon^b(k+1)]$, on which the best ε must exist,is

$$[\varepsilon^a(k+1),\ \varepsilon^b(k+1)] = [\varepsilon^u(k), \varepsilon^b(k)] \tag{11a}$$

if $\varepsilon^u(k) \precsim \varepsilon^v(k)$ and

$$[\varepsilon^a(k+1),\ \varepsilon^b(k+1)] = [\varepsilon^a(k), \varepsilon^v(k)] \tag{11b}$$

if $\varepsilon^u(k) \succsim \varepsilon^v(k)$.

These are the principle of iterative linear search methods. Especially the golden section method renews the section satisfying following conditions;

(a) In either case (i) or (ii), the width of new section is the same, that is

$$\varepsilon^b(k) - \varepsilon^u(k) = \varepsilon^v(k) - \varepsilon^a(k).$$

(b) When parameters $\varepsilon^u(k+1)$ and $\varepsilon^v(k+1)$ are picked up as

$$\varepsilon^a(k+1) < \varepsilon^u(k+1) < \varepsilon^v(k+1) < \varepsilon^b(k+1)$$

on the new section $[\varepsilon^a(k+1), \varepsilon^b(k+1)]$, one of them takes the same value with either $\varepsilon^u(k)$ or $\varepsilon^v(k)$ which ever is not used in the deduction of the section $[\varepsilon^a(k), \varepsilon^b(k)]$. That is to say, if $\varepsilon^u(k) \precsim \varepsilon^v(k)$ then $\varepsilon^u(k+1) = \varepsilon^v(k)$ and if $\varepsilon^u(k) \succsim \varepsilon^v(k)$ then $\varepsilon^v(k+1) = \varepsilon^u(k)$.

It is necessary and sufficient for the above two condition that

$$\varepsilon^u(k) = \varepsilon^a(k) + (1-\gamma)(\varepsilon^b(k) - \varepsilon^a(k)) \tag{12a}$$

$$\varepsilon^v(k) = \varepsilon^a(k) + \gamma(\varepsilon^b(k) - \varepsilon^a(k)), \tag{12b}$$

where γ is the golden section ratio $\gamma = 0.618034$.

For comparison of the two alternatives $(\varepsilon^u(k), \omega(\varepsilon^u(k))$ and $(\varepsilon^v(k), \omega(\varepsilon^v(k)))$ in the process of the golden section method, the simple majority decision rule is used. In other words, we ask odd numbers of decision-makers which of $(\varepsilon^u(k), \omega(\varepsilon^u(k)))$ and $(\varepsilon^v(k), \omega(\varepsilon^v(k)))$ is preferable, compare the number of decision-makers who prefer $(\varepsilon^u(k), \omega(\varepsilon^u(k)))$ to those who prefer $(\varepsilon^v(k), \omega(\varepsilon^v(k)))$, and designate the most prefered as the social choice. This interactive procedure is done at every renewed iteration. Under Assumption 2, weak order relation consists for the social preference between $\varepsilon^u(k)$ and $\varepsilon^v(k)$ by the simple majority decision rule. By use of this information, it is possible to carry out procedure in a renewal iteration of the golden section method. For the initial section of the golden section method $[\varepsilon^a(1), \varepsilon^b(1)]$, let

$$\varepsilon^a(1) = \varepsilon_{\min},\ \varepsilon^b(1) = \varepsilon_{\max},$$

where $\varepsilon_{\min}$ is as in (8) and $\varepsilon_{\max}$ as in (9a).

4. Computational Example

In order to assure that the golden section method with preference judgements of plural decision makers is available as a procedure to produce the social choice on bi-objective problems, numerial experiments on very simple examples were performed and the obtained results are reported in this section. To simplify the bi-objective problem (1), the constraint $\boldsymbol{x} \in X$ is not considered. Let the objective functions be

$$f_1(\boldsymbol{x}) = (x_1 - 2)^2 + (x_2 - 1)^2$$

and

$$f_2(\boldsymbol{x}) = (x_1 - 1)^2 + (x_2 - 2)^2.$$

The non-inferior value on $f_1 - f_2$ plane are analytically obtained as a curve

$$\sqrt{f_1} + \sqrt{f_2} = \sqrt{2} \qquad (f_1, f_2 \geqq 0).$$

Hence the ε-parameter's area E_a of the ε-constraint problem for this bi-objective problem is

$$E_a = \{\varepsilon \mid 0 \leqq \varepsilon \leqq 2\}. \tag{13}$$

In order to have the decision-makers' preferences for the objective function value $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))^T$, a pair of objective function values should be suggested to examinees and their preference judgements should be based on comparison of this pair. In this experiment, however, the forms of the decision-makers' preference functions were given as concrete functional forms on the $\boldsymbol{f}$-space and the decision-makers' preference judgements were determined from the function values, since the purpose of this experiment is to assure only the property of the algorithm and the convergence of the procedure. It was provided that the smaller the preference function value calculated from an objective function value is, the more preferable the objective function value is for a decision-maker. Actually let the number of decision-makers be five ($m = 5$) and each preference function be as follows;

$$\phi_1(\boldsymbol{f}) = (f_1 - 0.7)^2 + (f_2 - 1.7)^2$$

$$\phi_2(\boldsymbol{f}) = (f_1 - 2.0)^2 + (f_2 - 1.3)^2$$

$$\phi_3(\boldsymbol{f}) = (f_1 + 0.5)^2 + (f_2 - 0.8)^2$$

$$\phi_4(\boldsymbol{f}) = (f_1 - 0.75)^2 + (f_2 - 0.25)^2$$

$$\phi_5(\boldsymbol{f}) = (f_1 - 1.2)^2 + (f_2 + 0.8)^2$$

When the golden section method concerning ε was performed for problem (7), the upper limit and the lower limit of (13) were calculated and this interval was chosen to be the initial section $[\varepsilon^a(1), \varepsilon^b(1)]$, where $\varepsilon^a(1) = 7.4913 \times 10^{-7}$, $\varepsilon^b(1) = 2.0000$. The values of $\varepsilon^u(1)$ and $\varepsilon^v(1)$ satisfying

$$\varepsilon^a(1) < \varepsilon^u(1) < \varepsilon^v(1) < \varepsilon^b(1)$$

were calculated with the golden section ratio as $\varepsilon^u(1) = 7.6393 \times 10^{-1}$, $\varepsilon^v(1) = 1.2361$. By solving the ε-constraint problems for $\varepsilon = \varepsilon^u(1)$ and $\varepsilon = \varepsilon^v(1)$, two non-inferior values

$$\begin{aligned} \boldsymbol{f}^{\,u}(1) &= (\varepsilon^u(1), \omega(\varepsilon^u(1))) \\ &= (7.6393 \times 10^{-1}, 2.9259 \times 10^{-1}), \end{aligned}$$

$$\begin{aligned} \boldsymbol{f}^{\,v}(1) &= (\varepsilon^v(1), \omega(\varepsilon^v(1))) \\ &= (1.2361, 9.1984 \times 10^{-2}) \end{aligned}$$

were obtained corresponding to $\varepsilon^u(1)$ and $\varepsilon^v(1)$ respectively. Individual preferences of the five decision-makers for this pair of non-inferior values were determined from the pairs of function values $(\phi_i(\boldsymbol{f}^{\,u}(1)), \phi_i(\boldsymbol{f}^{\,v}(1))), i = 1, \cdots, 5$. The result is shown in Table 1. From this result, the social preference obtained by the simple majority decision rule was

$$\boldsymbol{f}^{\,u}(1) \succ \boldsymbol{f}^{\,v}(1).$$

The iterative process with the golden section method is shown in Table 2. Let the stop criterion be

$$| \varepsilon^a(k) - \varepsilon^b(k) | < 1.0 \times 10^{-3}$$

The non-inferior values corresponding to the convergent points were as follows;

$$\varepsilon^o = 7.7262 \times 10^{-1}, \boldsymbol{f}^{\,o} = (7.7262 \times 10^{-1}, 2.8724 \times 10^{-1})$$

In the experiment, the preference degrees were judged with square errors from an ideal point where a decision-maker's preference function is equal to zero. However it is impossible for men to judge details precisely. Therefore taking this into consideration, the errors' scale were transformed and the judgement was done with the integer part of the errors. For example, the first decision-maker's preference function

$$\phi_1(\boldsymbol{f}\,) = (f_1 - 0.7)^2 + (f_2 - 1.7)^2$$

is transformed into

$$\phi_1(\boldsymbol{f}\,) = \mathrm{INT}(((f_1 - 0.7)^2 + (f_2 - 1.7)^2) \times \boxed{10000}).$$

By exchanging the $\boxed{}$ part from 100 to 100,000 for five decision-makers' preference functions, the roughness was changed and experiments were done. The result is shown in Table 3. From this result it was found that the social choices were different depending on the roughness of judgements.

Table 1. The preference of the first iteration

decision-maker 1	$\boldsymbol{f}^{\,u}(1) \succ_1 \boldsymbol{f}^{\,v}(1)$
decision-maker 2	$\boldsymbol{f}^{\,u}(1) \prec_2 \boldsymbol{f}^{\,v}(1)$
decision-maker 3	$\boldsymbol{f}^{\,u}(1) \succ_3 \boldsymbol{f}^{\,v}(1)$
decision-maker 4	$\boldsymbol{f}^{\,u}(1) \succ_4 \boldsymbol{f}^{\,v}(1)$
decision-maker 5	$\boldsymbol{f}^{\,u}(1) \prec_5 \boldsymbol{f}^{\,v}(1)$
social preference	$\boldsymbol{f}^{\,u}(1) \succ \boldsymbol{f}^{\,v}(1)$

Table 2. Iterative Process

k	$\varepsilon^a(k)$	$\varepsilon^u(k)$	$\varepsilon^v(k)$	$\varepsilon^b(k)$
0	7.4913×10^{-7}	7.6393×10^{-1}	1.2361	2.0000
	$(7.4913 \times 10^{-7}, 1.9976)$	$(7.6393 \times 10^{-1}, 2.9258 \times 10^{-1})$	$(1.2361, 9.1984 \times 10^{-2})$	$(2.0000, 1.4609 \times 10^{-4})$
1	7.4913×10^{-7}	4.7214×10^{-1}	7.6393×10^{-1}	1.2361
	$(7.4913 \times 10^{-7}, 1.9976)$	$(4.7214 \times 10^{-1}, 5.2889 \times 10^{-1})$	$(7.6393 \times 10^{-1}, 2.9258 \times 10^{-1})$	$(1.2361, 9.1984 \times 10^{-2})$
2	4.7214×10^{-1}	7.6393×10^{-1}	9.4427×10^{-1}	1.2361
	$(4.7214 \times 10^{-1}, 5.2889 \times 10^{-1})$	$(7.6393 \times 10^{-1}, 2.9258 \times 10^{-1})$	$(9.4427 \times 10^{-1}, 1.9646 \times 10^{-1})$	$(1.2361, 9.1984 \times 10^{-2})$
3	4.7214×10^{-1}	6.5248×10^{-1}	7.6393×10^{-1}	9.4427×10^{-1}
	$(4.7214 \times 10^{-1}, 5.2889 \times 10^{-1})$	$(6.5248 \times 10^{-1}, 3.6865 \times 10^{-1})$	$(7.6393 \times 10^{-1}, 2.9258 \times 10^{-1})$	$(9.4427 \times 10^{-1}, 1.9646 \times 10^{-1})$
4	6.5248×10^{-1}	7.6393×10^{-1}	8.3282×10^{-1}	9.4427×10^{-1}
	$(6.5248 \times 10^{-1}, 3.6865 \times 10^{-1})$	$(7.6393 \times 10^{-1}, 2.9258 \times 10^{-1})$	$(8.3282 \times 10^{-1}, 2.5237 \times 10^{-1})$	$(9.4427 \times 10^{-1}, 1.9646 \times 10^{-1})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
14	7.7161×10^{-1}	7.7252×10^{-1}	7.7308×10^{-1}	7.7398×10^{-1}
	$(7.7161 \times 10^{-1}, 2.8786 \times 10^{-1})$	$(7.7252 \times 10^{-1}, 2.8731 \times 10^{-1})$	$(7.7308 \times 10^{-1}, 2.8697 \times 10^{-1})$	$(7.7398 \times 10^{-1}, 2.8642 \times 10^{-1})$
15	7.7161×10^{-1}	7.7217×10^{-1}	7.7252×10^{-1}	7.7308×10^{-1}
	$(7.7161 \times 10^{-1}, 2.8786 \times 10^{-1})$	$(7.7217 \times 10^{-1}, 2.8752 \times 10^{-1})$	$(7.7252 \times 10^{-1}, 2.8731 \times 10^{-1})$	$(7.7308 \times 10^{-1}, 2.8697 \times 10^{-1})$
16	7.7217×10^{-1}	7.7252×10^{-1}	7.7273×10^{-1}	7.7308×10^{-1}
	$(7.7217 \times 10^{-1}, 2.8752 \times 10^{-1})$	$(7.7252 \times 10^{-1}, 2.8731 \times 10^{-1})$	$(7.7273 \times 10^{-1}, 2.8717 \times 10^{-1})$	$(7.7308 \times 10^{-1}, 2.8697 \times 10^{-1})$

Table 3. The social choice in different ability of decision-makers

not transformed	$(7.7262 \times 10^{-1}, 2.8724 \times 10^{-1})$
$\times 100000$	$(7.7353 \times 10^{-1}, 2.8669 \times 10^{-1})$
$\times 10000$	$(8.1207 \times 10^{-1}, 2.8205 \times 10^{-1})$
$\times 1000$	$(8.0301 \times 10^{-1}, 2.8260 \times 10^{-1})$
$\times 100$	$(8.5692 \times 10^{-1}, 2.4536 \times 10^{-1})$

5. Conclusion

An algorithm was presented which provides a solution as a social choice, obtained by aggregating plural decision-makers' preferences on a non-inferior value set of a bi-objective problem. Especially in the case that the simple majority decision rule is used to construc a social preference from the individual preferences, it was shown that, under proper assumptions, the bi-objective problem can be transformed into an ε-parameter choice problem by using the ε-constraint method. Then it was suggested that the golden section method can be applied to this problem. An experiment for a simple problem was done and the availability of this method was proven.

There are certain issues that have not been considered in this paper. These include (1) applications to practical problems, (2) performance of actual preference judgements by decision-makers as examinees, (3) consideration of the vagueness of these performance judgements, (4) extension to general multi-objective problem, etc. are the unsettled theme.

Notes

♣Weak order relation: an order which satisfies following conditions is called the weak order. Here it is defined that 'x is preferable to y, or both are preferable at same degree' is represented as '$x \succsim y$'. Three conditions are represented as follows:

(1) Reflexive: $(x \succsim x)$ for $\forall x \in X$

(2) Connected: $(x \succsim y$ or $y \succsim x)$ for $\forall x, y \in X$

(3) Transitive: $\{(x \succsim y$ and $y \succsim x) \Rightarrow x \succsim z\}$ for $\forall x, y, z \in X$

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